2.6 Proof by contradiction

- A proof by contradiction or indirect proof starts by assuming that a theorem is false and then shows some logical inconsistency arises as a result of the assumption.

Suppose the theorem is: $(\forall a \forall b \ \sqrt{a} + \sqrt{b} \neq \sqrt{a+b})$

negate the theorem: $\neg(\forall a \forall b \ \sqrt{a} + \sqrt{b} \neq \sqrt{a+b})$
$\equiv \exists a \exists b \ (\sqrt{a} + \sqrt{b} = \sqrt{a+b})$

- Proof by contrapositive is a special case of proof by contradiction.
  - It only works for conditional statements

Different proofs for the theorem $p \to q$

Direct proof: Assume p. Follow a series of steps to conclude q.
Proof by contrapositive: Assume $\neg q$. Follow a series of steps to conclude $\neg p$.
Proof by contradiction: Assume $p \wedge \neg q$. Follow a series of steps to conclude $r \wedge \neg r$ for some proposition r.

2.7 Proof by Cases

- A proof by cases of a universal statement, such as $\forall x\, P(x)$, breaks the domain for variable x into different classes and gives a different proof for each class.
  - The proof for each class is called a case.
    - * Every value in domain must be included in at least one class.
      - △ A value can be in included in more than one case.

Example Prove that for every real number x, $x^2 \geq 0$

Case 1: $x < 0$

- Since $x < 0$, when we multiply both sides of the inequality $x < 0$ by x, the inequality is reversed, which results in $x^2 > 0$. This implies $x^2 \geq 0$.

Case 2: $x = 0$

- Multiply both sides of the equation $x = 0$ by x results in $x^2 = 0$, which implies that $x^2 \geq 0$

Case 3: $x > 0$

- Since $x > 0$, when we multiply both sides of the inequality $x > 0$ by x, the inequality stays in same direction, which results in $x^2 > 0$. This implies $x^2 \geq 0$.

The 3 cases cover all possibilities for x and each case has shown that $x^2 \geq 0$. Therefore, for every real number x, $x^2 \geq 0$.

- The term without loss of generality (abbreviated WLOG or w.l.o.g) is used in proofs to narrow the scope of a proof to one special case in situations when the proof can be easily adapted to the general case.

Example of theorem when you would use: For any two integers x and y, if x is even or y is even, then xy is even.

Proof: Without loss of generality assume x is even. Then ...

3.1 Sets and Subsets

- A set is a collection of objects. Objects may be of various types.
  - Each object in a set is called an element.

- Roster notation: list of all elements in curly braces
  - $A = \{2, 4, 6, 10\} = \{6, 10, 4, 2\} = \{2, 2, 6, 10, 4\}$

- "$\in$" indicates an element in a set, as in $2 \in A$.
- "$\notin$" indicates an element NOT in a set, as in $5 \notin A$

- Typically capital letters denote sets, and lowercase letters indicate an element or elements.
  - Example using the set A from above:

    $a \in A$, then a is equal to $2, 4, 6$, or 10.

- A set with no elements is called an empty set or null set.
  - Denoted by "$\emptyset$" or $\{\}$.

- The cardality of a finite set A, denoted by $|A|$, is the number of distinct elements in A.
  - Example: $A = \{2, 4, 6, 10\}$, then $|A| = 4$

$\mathbb{N} \rightarrow$ natural numbers is the set of all integers ≥ 0 $\quad 0, 1, 2, 3 \ldots$

$\mathbb{Z} \rightarrow$ integers, includes all integers $\quad \ldots -2, -1, 0, 1, 2 \ldots$

$\mathbb{Q} \rightarrow$ rational numbers, all real numbers that can be expressed $\frac{a}{b}$, $b \neq 0$ $\quad 0, \frac{1}{2}, 5.23, -\frac{5}{3}, 7$

$\mathbb{P} \rightarrow$ irrational numbers, all real numbers that cannot be expressed $\frac{a}{b}$, $b \neq 0$ $\quad \pi, \sqrt{2}, \frac{\sqrt{5}}{3}, -\frac{4}{\sqrt{6}}$

$\mathbb{R} \rightarrow$ real numbers $\quad 0, \frac{1}{2}, 5.23, -\frac{5}{3}, \pi, \sqrt{2}$

- The number 0 is niether positive nor negative
  - So $0 \notin \mathbb{Z}^+$ and $0 \notin \mathbb{Z}^-$

- Set builder notation
  - $A = \{x \in S : P(x)\}$
    - * "all x in S such that P(x)"

- The universal set usually denoted U, is a set that contains all elements mentioned in a particular context.
  - $\boxed{Ex}$ Set: Students in good academic standing at Notre Dame
    - Universal Set (U): All students at Notre Dame

- A is a subset of B if every element in A is also an element of B.
  - Denoted $A \subseteq B$
  - $A \not\subseteq B \rightarrow$ "A is NOT a subset of B."

- If $A \subseteq B$ and there is an element of B that is not an element of A, then A is a proper subset of B, denoted $A \subset B$